Symmetries of the Gowdy Equations and Spatial Topologies

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Abstract

We examine some kinds of discrete symmetries which are dynamically preserved, using the (generalized) Gowdy models of the first kind.

1 Introduction

It seems like there are some connections between dynamical properties of a spacetime and its spatial topology. For example, we know the recollapsing conjecture [1, 2, 3]; As is well known, in a positive curvature (topologically S^3) homogeneous and isotropic cosmological model, the universe contracts and recollapses after an expanding era. The conjecture claims that this recollapsing property does not depend on the symmetry imposed on the model, that is, any spacetime recollapses if the space is *topologically* S^3 (and if only appropriate energy and pressure conditions are fulfilled). Thus we may think of the recollapsing property as a result of the spatial topology. (For another interesting examples related to asymptotic dynamics, see [4, 5].)

In this article we compare some dynamical properties of the (generalized) Gowdy models of the first kind. The spatial manifold of this kind is a T^2-bundle over S^1, and each T^2-fiber is generated by two commuting local Killing vectors. (Such a model was first considered by Rendall [6], and discussed extensively by the author [7, 8]. These models are generalizations of Gowdy's T^3 model [9].) There are infinite number of such bundles which are topologically distinct from each other. However, since each fiber is generated by (local) Killing vectors in our models the natural reduced (spatial) manifold is S^1 in all cases. Because of this reason we can compare the dynamical properties of each model in rather detail.

What we investigate concerns symmetries or sets of symmetric data which are preserved temporarily. More specifically, we consider reflection symmetries (in a generalized sense). For example, for the usual T^3 Gowdy model any set of initial data which are described by even (spatial) functions evolves in such a way that the data remain even functions at any time. Also, the same statement is true even if we replace "even" by "odd", and in general we will see there is much larger set of such "reflection" symmetries which are preserved. We can consider corresponding symmetric data for any other models, but the symmetries are not necessarily preserved. We may interpret this is a manifestation of the influence of spatial topology.

This article basically treats the same topic as Ref.[8], but the presentation is made in a somewhat different way.

2 The Gowdy Models of the 1st Kind

The metric of the Gowdy models can be written in the form

$$ds^2 = -e^A(d\tau^2 - dx^2) + R[e^P(dy + Qdz)^2 + e^{-P}dz^2], \tag{1}$$

where A, R, P, and Q are functions of τ and x. There are two commuting Killing vectors for this metric, $\partial/\partial y$ and $\partial/\partial z$. We may think of this metric as that on $\mathbf{R}^4$, but we can also compactify the spatial part $\mathbf{R}^3$ to a T^2-bundle over S^1 if imposing appropriate boundary conditions on the metric functions. In such a case, the model is called a (generalized) Gowdy model of the 1st kind, which we will consider. After the compactification the two Killing vectors, in general, descend to local Killing vectors, generating the T^2-fibers.

Note that each T^2-fiber is characterized by its volume *and* two moduli parameters $(X, Y) \equiv (Q, e^{-P})$.

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3 Symmetries

As well known [9, 8], the vacuum Einstein equation for metric function R is given by the simple wave equation $(\partial_{\tau\tau} - \partial_{xx})R = 0$, and we can set without loss of generality, $R = \tau$. With this choice, the equations for functions P and Q are found to form a closed set of equations, and the remaining function A is integrable once P and Q are solved. So, we are led to concentrate on the moduli parameters X and Y, and the equations for them are derivable from the following (reduced) Hamiltonian H,

$$H = \int \mathcal{H} dx, \tag{2}$$

$$\mathcal{H} = \frac{1}{2}(\mathrm{g}^{AB}\Pi_A\Pi_B + e^{-2\tau}\mathrm{g}_{AB}X^{A\prime}X^{B\prime}), \tag{3}$$

where scripts $A, B, \cdots$ run 1 to 2. We have set $(X^1, X^2) \equiv (X, Y)$, and (Π_1, Π_2) is the set of the conjugate momenta. The "metric" g_{AB} on the moduli space $\mathcal{M}$ (spanned by X and Y) is given by the following *hyperbolic metric*

$$dS^2 = \frac{dX^2 + dY^2}{Y^2}. \tag{4}$$

So, the moduli space is also a hyperbolic plane H^2 with three dimensional isometry group $\mathrm{Isom}H^2$, of which connected component is $SL_2\mathbf{R}$. (Exactly speaking, $\mathcal{M}$ is not H^2 itself, but a quotient space of it. This point is explained in §.5.)

Note that the data for a moment (τ = constant), i.e., the configuration of the spatial manifold, is given by a loop $l(x)$ in the moduli space:

$$l : [0, 1] \longrightarrow \mathcal{M}, \quad l(0) = l(1), \tag{5}$$

and the whole spacetime is described by a one-parameter (τ) family of such loops: $l(x, \tau)$.

The isometry group $\mathrm{Isom}H^2$ provides symmetries of motion, i.e., if the "spacetime" $l(x, \tau)$ is a solution for the vacuum Einstein equation, then the action $g \cdot l(x, \tau)$ is also another solution for $g \in \mathrm{Isom}H^2$. This is one of well-known properties of the Gowdy equations.

We also should point out a rather trivial but important symmetry, which is given by the following reparametrization:

$$R \cdot l(x, \tau) = l(-x, \tau). \tag{6}$$

As is easily seen, R inverts (or reflects) each spatial configuration about the origin $x = 0$. This also define a symmetry of motion.

4 The Reflection Operators

While we may think of the operator R itself as a reflection operator, it is also possible to generalize reflections using the isometries $\mathrm{Isom}H^2$, i.e., we consider the composition

$$\mathcal{R} = R \cdot g, \quad g \in \mathrm{Isom}H^2. \tag{7}$$

Any such an operator defines a symmetry of motion. We should furthermore impose the condition for $\mathcal{R}$ to form a $\mathbf{Z}_2$ group (together with the identity). For convenience, defining $\mathcal{R}$ as a complex function

$$\mathcal{R} : \ z = X + iY \rightarrow \mathcal{R}(z), \tag{8}$$

this condition is expressed by

$$\mathcal{R}^2(z) = z, \quad \text{for all } z \in \mathbf{C}. \tag{9}$$

From this we find two classes of operators; one is given by R itself

$$\mathcal{R}_I \equiv R, \tag{10}$$

and the other is given by a one-parameter family

$$\mathcal{R}_\theta \equiv R \cdot \gamma \cdot f_\theta, \quad \theta \in [0, \pi), \tag{11}$$

where

$$\gamma(z) = -\bar{z}, \tag{12}$$

$$f_\theta(z) = \frac{z\cos 2\theta - \sin 2\theta}{z\sin 2\theta + \cos 2\theta}. \tag{13}$$

However, these operators do not necessarily preserve the boundary condition imposed on the solution $X(x,\tau)$ and $Y(x,\tau)$, so not all these operators are appropriate for a given spatial topology. To account for this point for more detail, we should explain the four representative models, called the T^3, E^3, Nil, and Sol models.

5 The Four Representative Models

We have noticed any spatial configuration is represented by a loop l on the moduli space $\mathcal{M}$, but this does not mean the metric functions $X(x)$ and $Y(x)$ are periodic with respect to the spatial coordinate x. This is because $\mathcal{M}$ is a quotient space. It is a classical fact that $\mathcal{M}$ is represented by H^2 *modulo* $SL_2\mathbf{Z}$, $\mathcal{M} \simeq H^2/SL_2\mathbf{Z}$, where $SL_2\mathbf{Z}$ is the mapping class group for T^2. (This discrete group, $SL_2\mathbf{Z}$, is generated by the shift $T: z \to z+1$ and the inversion $S: z \to -1/z$.) H^2 is naturally represented by the upper half plane $\mathbf{C}^+ \equiv \{z \in \mathbf{C} | \mathrm{Im}(z) > 0\}$, and a fundamental domain $\mathcal{F}$ for $\mathcal{M}$ is given by $\mathcal{F} = \{z \in \mathbf{C}^+ | -\frac{1}{2} \le \mathrm{Re}(z) \le \frac{1}{2}, |z| \ge 1\}$. See Fig.1. For the boundary, the two vertical lines (at $\mathrm{Re}(z) = \pm\frac{1}{2}$) should be identified, and the arc (at $|z| \le 1$) is folded about $z = i$ and the opposite sides are identified, as well. So, the point $z = i$ becomes singular. (These are a standard fact about the geometric structure on T^2. See a standard text for a detail.) Let us denote a curve on $\mathbf{C}^+$ which descends to a loop l on $\mathcal{M}$, as $\tilde{l}$. It is $\tilde{l}$ that has a direct correspondence to the boundary condition imposed on $X(x)$ and $Y(x)$. Apparently, $\tilde{l}$ is not necessarily a loop, so what is said above comes. Note that since two (topologically) distinct T^2-bundle over S^1 can have the same configuration of moduli parameters, correspondences of a loop l (or curve $\tilde{l}$) to a topology are not unique, but there are not ambiguities between the four models.

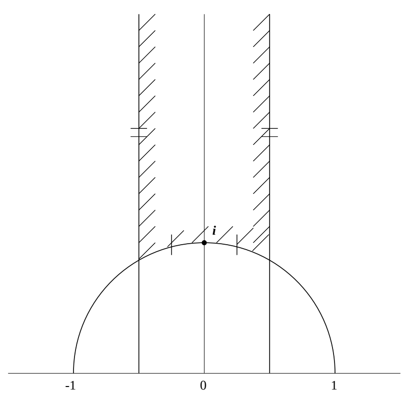

Figure 1: The fundamental domain $\mathcal{F}$ for the moduli space $\mathcal{M}$ is shown (the shaded region).

Now, we are in a position to describe the four models. (However, the following descriptions are not complete enough. See Ref.[8] for more complete ones, though they are less pictorial.)

(i) T^3 **model**: This model is the conventional one, for which the spatial manifold is topologically T^3. The boundary condition for the metric functions (moduli parameters) is simply given by the periodic boundary condition. A loop l for this model is also represented by a loop $\tilde{l}$ on $\mathbf{C}^+$. See Fig.2 (i). $\tilde{l}$ does not encircle the singular point $z = i$, so can smoothly contract to a point, which corresponds to a flat homogeneous space (Bianchi I space).

(ii) E^3 **model**: This model is the representative model for the models whose spatial part is T^3, $T^3/\mathbf{Z}_2$, $T^3/\mathbf{Z}_3$, $T^3/\mathbf{Z}_4$, or $T^3/\mathbf{Z}_6$. The boundary condition is periodic, but the loop $\tilde{l}$ on $\mathbf{C}^+$ must encircle the

singular point $z = i$, so *cannot* smoothly contract to a point. See Fig.2 (ii). The locally homogeneous limit corresponds to a Bianchi VII_0 space.

(iii) Nil model: $\tilde{l}$ is an open curve which is invariant under the shifting T. One end approaches to $\mathrm{Re}(z) = +\infty$, and the other to $\mathrm{Re}(z) = -\infty$. See Fig.2 (iii). $\tilde{l}$ descends to a loop on $\mathcal{M}$ which winds around the singular point $z = i$ only once. At a locally homogeneous limit, $\tilde{l}$ becomes a horizontal straight line, which corresponds to a Bianchi II space.

(iv) Sol model: $\tilde{l}$ is an open (arc-like) curve for which both ends approach to $\mathrm{Im}(z) = 0$; one end to $|z| < 1$ and the other end to $|z| > 1$. See Fig.2 (iv). $\tilde{l}$ descends to a loop on $\mathcal{M}$ which winds around the singular point $z = i$ twice or more. The locally homogeneous limit corresponds to a Bianchi VI_0 space.

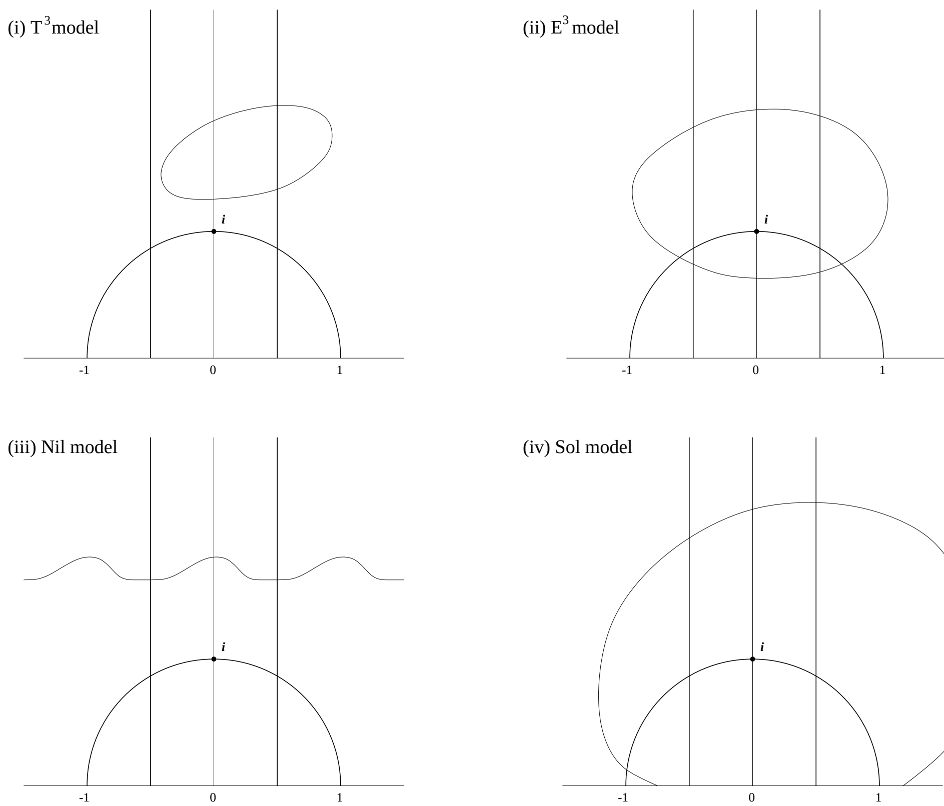

Figure 2: A morphological classification of the four models is shown. A possible curve $\tilde{l}$ is depicted for each model in the upper half plane $\mathbf{C}^+$. Every curve descends to a loop (in $\mathcal{M}$) after identifications generated by T and S.

6 The Invariant Sets

Table 1 shows the appropriate reflection operators for each model [8]. These operators are extracted by requiring they preserve the appropriate boundary condition for each model, i.e., $\tilde{l}$ belonging to a model be mapped by one of these operators to another $\tilde{l}'$ belonging to the same model. (More precisely, to get these reflection operators, we also require a compatibility with a translation operator [8].)

Note that once a reflection operator $\mathcal{R}$ is given, we can accordingly define a set of reflection symmetric solutions $\mathcal{S}(\mathcal{R})$:

$$\mathcal{S}(\mathcal{R}) \equiv \{l(x,\tau)|\ \mathcal{R}(l(x,\tau)) = l(x,\tau)\}. \tag{14}$$

These solutions are invariant under the action of the reflection operator. We call such a set of solutions an *invariant (sub)set.*

Model	Reflections
T^3	$\mathcal{R}_I$, $\mathcal{R}_\theta$
E^3	$\mathcal{R}_\theta$
Nil	$\mathcal{R}_0$
Sol	$\mathcal{R}_{\pi/4}$

Table 1: This shows which reflection operators are eligible for each model. The parameter θ in T^3 or E^3 model can take any value in the range $[0, \pi)$, whereas Nil and Sol models can take $\theta = 0$ and $\pi/4$, respectively. $\mathcal{R}_I$ is possible only for T^3 model.

Apparently, we have the largest union of such sets for the T^3 model, while that of the Nil or Sol model is the smallest. *This is a character coming from topological distinctions.*

7 Dynamical Interpretation

We interpret the above fact as follows.

Note first that the phase space $\mathcal{P}$ for the T^3 model is given by the space of the sets of four (smooth) periodic functions (X, Y, Π_X, Π_Y). The phase space for Nil or Sol model would of course be given by another space, since the metric functions for them are not periodic functions. However, we know it must still be possible to identify it with $\mathcal{P}$, since the base spaces of the bundles are all S^1. All we have to do to do this is to re-represent the metric functions in such a way that they represent deviations from the configuration of a locally homogeneous limit. Practically, this can be easily done by replacing the coordinate basis of the metric (1) by an invariant basis for the appropriate Bianchi type, which was given in §.5.

Next consider the reflection operators $\mathcal{R}_I$ and $\mathcal{R}_\theta$. Although they have been defined so they act on a configuration space, we can naturally extend them to act on the phase space using "Hamiltonian equations" $\Pi_X = Y^{-2}\dot{X}$, $\Pi_Y = Y^{-2}\dot{Y}$. Then those operators define a set of reflection symmetric data in $\mathcal{P}$ by requiring these data be invariant under the action of one of the operators. These sets are common for all the models, since the above two (Hamiltonian) equations are common even if the variables are re-represented in the way described above.

So, we have the same (underlying) phase space and the same reflection symmetric data sets for all the four models. The Hamiltonians are, however, different in this setting, and we can compare dynamical properties of the four models from this point of view. In fact we can interpret [8] the result of the previous section as telling that *only the reflection symmetries shown in Table.1 are preserved by the Hamiltonian flow.* This is a manifestation of the influence of topology to dynamics.

Remark: One might think that there were much more possibilities for another set of reflection symmetric data. However, since we are interested only in those preserved dynamically, it is sufficient to consider the operators $\mathcal{R}_I$ and $\mathcal{R}_\theta$.

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